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Design of IIR Filters by Bilinear transformation method :

Consider an analog filter with system function as

$$H_a(s) = \frac{b}{s+a} \quad \text{——} \ (1)$$

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a} \quad , \quad sY(s) + aY(s) = bX(s)$$

Taking inverse laplace transform,

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Integrating above equation betn. limits $nT-T$ & nT

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} y(t)\,dt = b\int_{nT-T}^{nT} x(t)\,dt$$

The trapezoidal rule for numeric integration is

$$\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}\left[a(nT) + a(nT-T)\right]$$

$$\therefore \ y(nT) - y(nT-T) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T)$$
$$= \frac{bT}{2}x(nT) + \frac{bT}{2}x(nT-T)$$

Put $y(n) = y(nT)$, $y(n-1) = y(nT-T)$

$$y(n)\left(1+\frac{aT}{2}\right) - y(n-1)\left(1-\frac{aT}{2}\right) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

Taking z- transform,

$$\left(1+\frac{aT}{2}\right)Y(z) - \left(1-\frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}(1+z^{-1})X(z)$$

the system function of digital filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}(1+z^{-1})}{1+\frac{aT}{2} - z^{-1}\left(1-\frac{aT}{2}\right)} = \frac{\frac{bT}{2}(1+z^{-1})}{(1-z^{-1})+\frac{aT}{2}(1+z^{-1})}$$

Dividing num & denom by $\frac{T}{2}(1+z^{-1})$

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a} \quad — (2)$$

Comparing eqns (1) & (2)

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

Let $z = re^{j\omega}$, $s = \sigma + j\Omega$

$$s = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right) = \frac{2}{T}\left(\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right)$$

$$= \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]\left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{(r\cos\omega+1)^2 + r^2\sin^2\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

Comparing with $s = \sigma + j\Omega$

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right]$$

$$\Omega = \frac{2}{T}\cdot\frac{2r\sin\omega}{1+r^2+2r\cos\omega}$$

Case 1: If $r<1$

$$\sigma = \frac{2}{T}\,\frac{r^2-1}{1+r^2+2r\cos\omega}$$

$\sigma<0$

Left half of s-plane maps onto points inside the unit circle in z-plane.

Case 2: When $r>1$

$\sigma>0$

Right half of s-plane maps onto points outside the unit circle in z-plane.

Case 3: When $r=1$, $\sigma=0$ → marginally stable condition

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right]$$

$$\Omega = \frac{2}{T}\cdot\frac{2r\sin\omega}{1+r^2+2r\cos\omega}$$

If $r=1$

$$\Omega = \frac{2}{T}\left(\frac{2\sin\omega}{2+2\cos\omega}\right) = \frac{2}{T}\left[\frac{2\sin\omega/2\,\cos\omega/2}{\cos^2\omega/2+\sin^2\omega/2 + \cos^2\frac{\omega}{2}-\sin^2\frac{\omega}{2}}\right]$$

$$\sin\theta = 2\sin\theta/2\,\cos\theta/2$$

$$1 = \cos^2\theta/2 + \sin^2\theta/2$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$\Omega = \frac{2}{T}\cdot\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\omega/2} = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\Omega = \frac{2}{T} \tan\frac{\omega}{2}$$

$\omega = \Omega T$

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

→ There is a non linear relation between analog frequency Ω and digital frequency ω.

→ $$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

ω

$\frac{\omega}{2}$

$\tan\frac{\omega}{2} \rightarrow \frac{\omega}{2}$

For small θ, $\tan\theta \simeq \theta$

$\therefore$ For small ω,

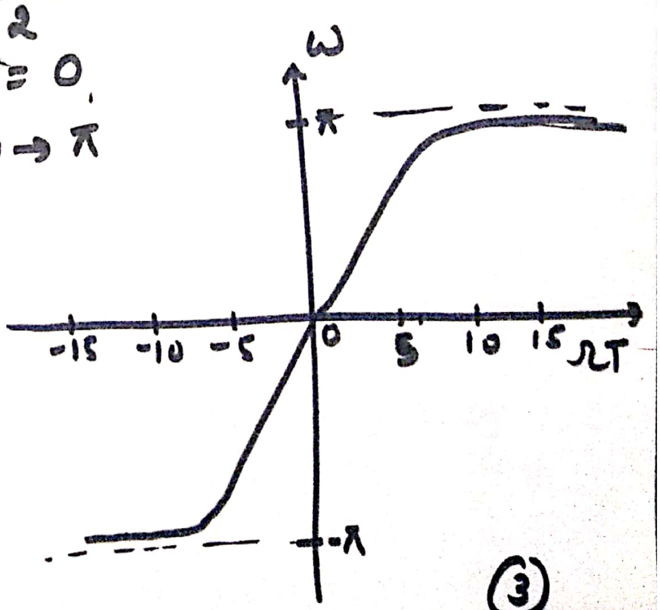
$$\Omega = \frac{2}{T}\cdot\frac{\omega}{2} = \frac{\omega}{T} \Rightarrow \underline{\omega = \Omega T}$$

For <u>low frequencies</u>, the relationship is linear

→ $$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

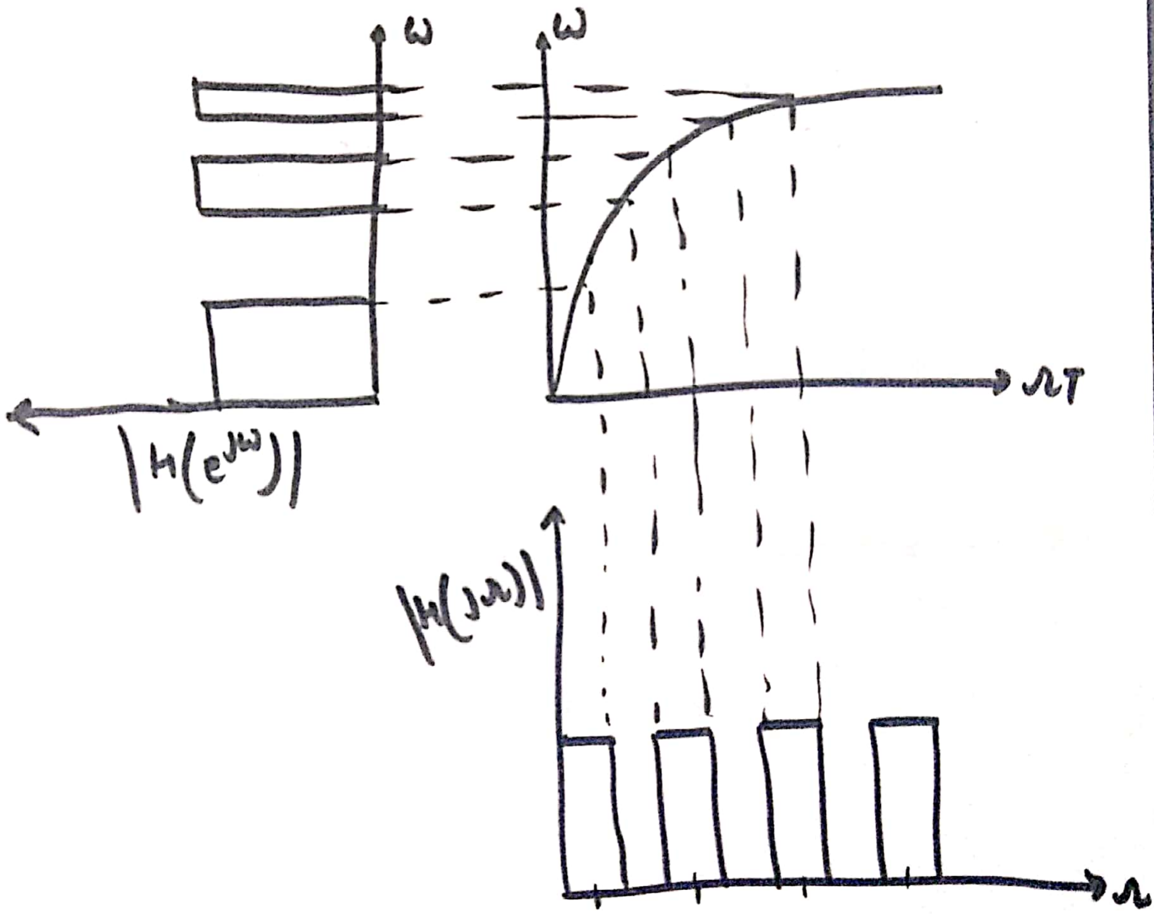
If $\Omega = 0$, $\omega = 0$.

$\Omega \rightarrow \infty$, $\omega \rightarrow \pi$



→ For higher frequencies, the mapping is non linear

→ The lower frequencies in analog domain are expanded in digital domain, whereas the higher frequencies are compressed. This effect of nonlinear mapping is known as <u>frequency warping effect</u>.



→ Non linear compression of frequency axis.

Bilinear Transformation method :-

$$\frac{b}{s+a} \longrightarrow \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

Ques Convert the analog filter with system function $H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$ into a digital IIR filter by using Bilinear trans-formation method. The digital filter is to have a resonant freq of $\omega_r = \frac{\pi}{2}$.

Sol - Poles : $(s+0.1)^2+16=0$

$$(s+0.1)^2 - (j4)^2 = 0$$

$$\Rightarrow (s+0.1-j4)(s+0.1+j4)=0$$

$$s = -0.1 \pm j4 = \sigma \pm j\Omega$$

$$\Omega_r = 4 \quad , \quad \omega_r = \frac{\pi}{2} \text{ (Given)}$$

$$\Omega_r = \frac{2}{T}\tan\frac{\omega_r}{2}$$

$$\Rightarrow T = \frac{2}{\Omega_r}\tan\frac{\omega_r}{2} = \frac{2}{4}\tan\frac{\pi}{4} = \frac{1}{2}$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{\frac{4(1-z^{-1})}{1+z^{-1}} + 0.1}{16\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.01 + 0.2\left(4\frac{1-z^{-1}}{1+z^{-1}}\right) + 16}$$

$$H(z) = \frac{4.1 + 0.2z^{-1} - 3.9z^{-2}}{32.81 - 0.02z^{-1} + 31.21z^{-2}}$$

$$H(z) = \frac{0.128 + 0.0061z^{-1} - 0.1189z^{-2}}{1 + 0.0006z^{-1} + 0.9512z^{-2}}$$

Ideal Low pass filter:

Filters are specified by

(a) Frequency specifications : Include passband edge frequency ω_p and stop band edge frequency ω_s.

(b) Attenuation specifications : include passband attenuation (or gain) 'Ap' and stop band attenuation (or gain) 'A_s'

$$\text{Gain } G = |H(e^{j\omega})|$$

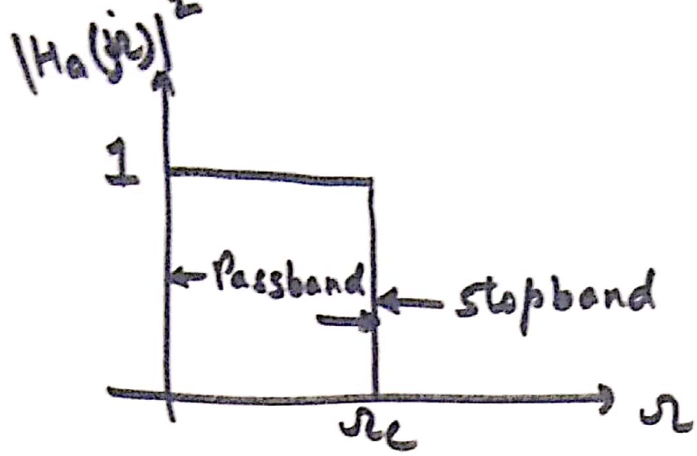
$$\text{Attenuation } A = \frac{1}{G} = \frac{1}{|H(e^{j\omega})|}$$

$$G_{dB} = 20\log_{10} G = 20\log_{10}|H(e^{j\omega})|$$

$$A_{dB} = 20\log_{10}\left(\frac{1}{G}\right) = -20\log_{10}|H(e^{j\omega})|$$

In Passband $[0, \omega_p] \rightarrow$ attenuation specification is smallest

In stop band $[\omega_s, \pi] \rightarrow$ attenuation specification is greatest



$\Omega \leq \Omega_c \rightarrow$ Passband of filter
$\Omega > \Omega_c \rightarrow$ stopband of filter
$\Omega_c \rightarrow$ cut off frequency.

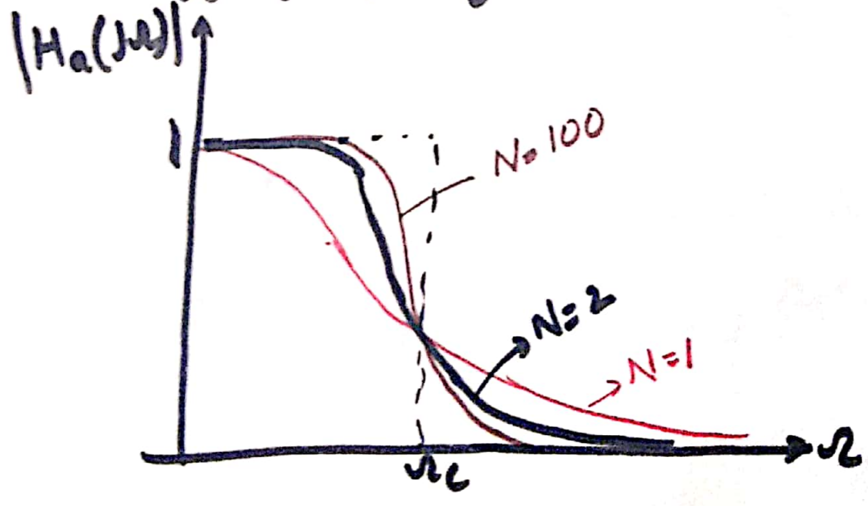
Butterworth low pass filter:

The butterworth LPF has a magnitude response given by

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{0.5}}$$

$N \rightarrow$ order of filter
$\Omega_c \rightarrow$ cut off frequency (rad/sec)

$\rightarrow$

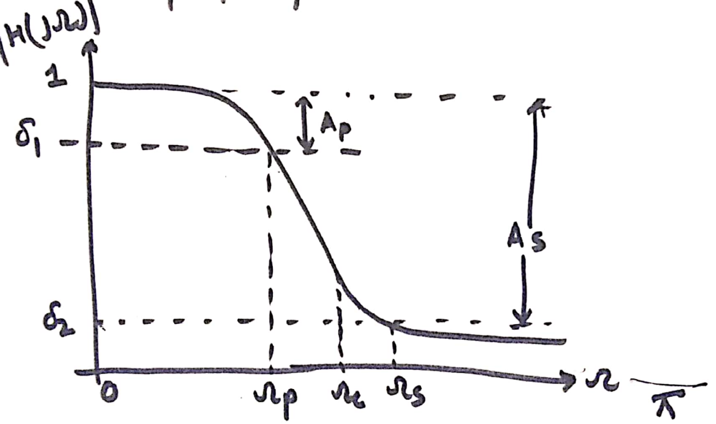


The magnitude response has maximally flat passband and stopband.

→ By increasing the filter order N, the response approximates the ideal response.

$$|H_a(j0)| = 1 \quad , \quad |H_a(j\Omega_c)| = \frac{1}{\sqrt{2}}$$

$$|H_a(j\infty)| = 0$$



$\delta_1 = \frac{1}{\sqrt{1+\varepsilon^2}}$ $\qquad$ ε → Passband ripple parameter

$\delta_2 = \frac{1}{A}$ $\qquad$ A → stopband attenuation parameter

- $A_p = -20\log_{10}\delta_1$ $\quad$ | $\quad$ $\delta_1 \le |H(j\Omega)| \le 1, \quad 0 \le \Omega \le \Omega_p$
- $A_s = -20\log_{10}\delta_2$ $\quad$ | $\quad$ $|H(j\Omega)| \le \delta_2, \quad \Omega_s \le \Omega \le \pi$

2. Design a single pole low pass digital filter with a 3dB Bandwidth of (3 dB cut off frequency) $\omega_c = 0.2\pi$, by applying bilinear transformation to the analog butter-worth analog filter $H(s) = \frac{\Omega_c}{s+\Omega_c}$

Ω_c is cut off frequency of analog filter

Sol $\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = \frac{2}{T}\tan 0.1\pi = \frac{0.65}{T}$

$$H(s) = \frac{\frac{0.65}{T}}{s + \frac{0.65}{T}}$$

$$H(z) = \frac{\frac{0.65}{T}}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + \frac{0.65}{T}}$$

$$H(z) = \frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$$